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Contractions weakly similar to unitaries. II

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In this paper we continue the study of contractions, weakly similar to unitaries, begun in [9]. Here we consider the case, when the characteristic function is not isometric a.e. on the unit circle, and prove the reflexivity of such contractions under a general assumption. Our paper is organized as follows. After giving the necessary definitions and notations in Section 0, we introduce the notion of weak similarity in Section 1. Our main result is proved in Section 2, while in Section 3 we make some concluding remarks. The theory of contractions, elaborated by B. SZ.-NAGY and C. FOIAŞ will be applied, the main reference is their monograph [12].

0. Definitions and notations

If $\mathfrak{H}$ is a (complex, separable) Hilbert space, then $\mathscr{L}(\mathfrak{H})$ denotes the set of all (bounded, linear) operators acting on $\mathfrak{H}$. For an arbitrary subset $\mathscr{A}\subset\mathscr{L}(\mathfrak{H})$, Lat $\mathscr{A}$ stands for the lattice of invariant subspaces of $\mathscr{A}$, while for an arbitrary set S of (closed) subspaces of $\mathfrak{H}$, $\operatorname{Alg} S$ is the algebra of operators which leave invariant each element of S. A subalgebra $\mathscr{A}\subset\mathscr{L}(\mathfrak{H})$ is called reflexive, if $\operatorname{Alg}\operatorname{Lat}\mathscr{A}=\mathscr{A}$ (cf. [5]).

For an operator $T\in\mathscr{L}(\mathfrak{H})$, $\operatorname{Alg} T$ denotes the weakly closed algebra generated by T and the identity. It is clear that $\operatorname{Lat} T=\operatorname{Lat}\operatorname{Alg} T$. T is called reflexive, if $\operatorname{Alg} T$ is reflexive, i.e. $\operatorname{Alg}\operatorname{Lat} T=\operatorname{Alg} T$. $\{T\}'$ and $\{T\}''$ denote the commutant and bicommutant of T, respectively, and $\operatorname{Lat}'' T:=\operatorname{Lat}\{T\}''$, $\operatorname{Hyplat} T:=\operatorname{Lat}\{T\}'$. If T is a completely non-unitary (c.n.u.) contraction, then

$$H^\infty(T):=\{w(T):\ w\in H^\infty\},$$

where H^∞ denotes the Hardy class of bounded analytic functions, and the Sz.-Nagy, Foiaş functional calculus is applied for T.

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The contraction $T\in\mathscr{L}(\mathfrak{H})$ belongs to the class C_{11} or C_{10}, if for every non-zero vector $h\in\mathfrak{H}$ we have

$$\lim_{n\to\infty}\|T^n h\|\neq 0\neq\lim_{n\to\infty}\|T^{*n}h\|,$$

or

$$\lim_{n\to\infty}\|T^n h\|\neq 0=\lim_{n\to\infty}\|T^{*n}h\|,$$

respectively. If T is a C_{11}-contraction, then

$$\mathrm{Lat}_1 T:=\{\mathfrak{M}\in\mathrm{Lat}\,T:\ T|\mathfrak{M}\in C_{11}\}$$

is a lattice under set-inclusion as partial ordering, in which the greatest lower bound "$\overset{(1)}{\cap}$" is generally different from the intersection "$\cap$". $\mathrm{Hyplat}_1 T:=\mathrm{Lat}_1 T\cap\cap\mathrm{Hyplat}\,T$ is a sublattice of $\mathrm{Lat}_1 T$ (cf. [8]).

D will denote the open unit disc of the complex plane, C its boundary, and m the normalized Lebesgue measure on C. For a contraction $T\in\mathscr{L}(\mathfrak{H})$, $\mathfrak{D}_T:=((I-T^*T)\mathfrak{H})^-$ and $\mathfrak{D}_{T*}:=((I-TT^*)\mathfrak{H})^-$ denote its defect spaces, and $\{\Theta_T(\lambda),\mathfrak{D}_T,\mathfrak{D}_{T*}\}$ its characteristic function in the sense of Sz.-Nagy and Foiaş, i.e. λ alters in **D** and $\Theta_T(\lambda)\in\mathscr{L}(\mathfrak{D}_T,\mathfrak{D}_{T*})$ is defined by

$$\Theta_T(\lambda)=[-T+\lambda(I-TT^*)^{1/2}(I-\lambda T^*)^{-1}(I-T^*T)^{1/2}]\,|\,\mathfrak{D}_T.$$

Moreover, Δ_T stands for the operator-valued function defined on C by the formula

$$\Delta_T(e^{it})=[I-\Theta_T(e^{it})^*\Theta_T(e^{it})]^{1/2}.$$

(Θ_T has radial limit a.e. on C.)

If $T\in\mathscr{L}(\mathfrak{H})$, $S\in\mathscr{L}(\mathfrak{K})$, then $\mathscr{I}(T,S)$ denotes the set of intertwining operators:

$$\mathscr{I}(T,S)=\{X\in\mathscr{L}(\mathfrak{H},\mathfrak{K}):\ XT=SX\}.$$

We say that T can be injected into S, and write $T\overset{i}{\prec}S$, if $\mathscr{I}(T,S)$ contains an injection; T is a quasi-affine transform of S, if $\mathscr{I}(T,S)$ contains a quasi-affinity, i.e. an injection with dense range; and T, S are quasi-similar, if they are quasi-affine transforms of each other.

A system $\{\mathfrak{H}_n\}_n$ of subspaces of $\mathfrak{H}$ is called to be basic, if $\mathfrak{H}_n\dotplus(\bigvee_{k\neq n}\mathfrak{H}_k)=\mathfrak{H}$, for every n, and $\bigcap_n(\bigvee_{k\geq n}\mathfrak{H}_k)=\{0\}$ (cf. [1]).

1. Weak similarity

We begin by introducing the notion of contractions, weakly similar to unitaries, in a bit more general setting than in [9]. Namely, we give the following

Definition 1. The operators $T\in\mathcal{L}(\mathfrak{H})$ and $S\in\mathcal{L}(\mathfrak{K})$ are called *weakly similar,* if there exist basic systems $\{\mathfrak{H}_n\}_n$ and $\{\mathfrak{K}_n\}_n$ in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, such that $\mathfrak{H}_n\in\operatorname{Hyplat} T$, $\mathfrak{K}_n\in\operatorname{Hyplat} S$, and $T|\mathfrak{H}_n$ is similar to $S|\mathfrak{K}_n$, for every n.

$T\in\mathcal{L}(\mathfrak{H})$ is *weakly similar to unitary,* if T is weakly similar to a unitary operator.

Remark 2. Weak similarity is clearly a weaker relation than similarity, but stronger than quasi-similarity. In fact, let P_n and Q_n denote the projections onto the subspaces $\mathfrak{H}_n$ and $\mathfrak{K}_n$ with respect to the decompositions $\mathfrak{H}=\mathfrak{H}_n\dotplus(\bigvee_{k\neq n}\mathfrak{H}_k)$ and $\mathfrak{K}=\mathfrak{K}_n\dotplus(\bigvee_{k\neq n}\mathfrak{K}_k)$, respectively. Now, choosing intertwining affinities $A_n\in\mathcal{I}(T|\mathfrak{H}_n, S|\mathfrak{K}_n)$, for every n, and sequences $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ of positive numbers such that

$$\sum_n \alpha_n\|A_n\|\,\|P_n\|<\infty \quad\text{and}\quad \sum_n \beta_n\|A_n^{-1}\|\,\|Q_n\|<\infty,$$

we can define intertwining quasi-affinities $X\in\mathcal{I}(T,S)$ and $Y\in\mathcal{I}(S,T)$ by the equations

$$Xf=\sum_n \alpha_n A_n P_n f \quad (f\in\mathfrak{H}) \quad\text{and}\quad Yg=\sum_n \beta_n A_n^{-1}Q_n g \quad (g\in\mathfrak{K}).$$

The operator occurring in [9, Proposition 2] provides an example for a C_{11}-contraction which is not weakly similar to unitary. Since every C_{11}-contraction is quasi-similar to a unitary operator, we obtain that weak similarity is an actually stronger relation than quasi-similarity in the class of C_{11}-contractions. (Quasi-similarity was characterized in the class C_{11} in terms of decomposibility by C. Apostol [1].)

Remark 3. In [9] a contraction $T\in\mathcal{L}(\mathfrak{H})$ is called weakly similar to unitary, if there exists a basic system $\{\mathfrak{H}_n\}_n$ consisting of hyperinvariant subspaces of T such that $T|\mathfrak{H}_n$ is similar to a unitary operator $U_n\in\mathcal{L}(\mathfrak{K}_n)$, for every n. However, we can define a unitary operator U acting on the space $\mathfrak{K}=\bigoplus_n\mathfrak{K}_n$ as the orthogonal sum $U=\bigoplus_n U_n$. Constructing an intertwining quasi-affinity $X\in\mathcal{I}(T,U)$ as in the preceding remark, an application of [8, Proposition 6] shows that $\mathfrak{K}_n=(X\mathfrak{H}_n)^-\in\operatorname{Hyplat} U$, for every n. Therefore, T is weakly similar to U, i.e. T is weakly similar to unitary in the sense of our present definition too. Hence the two definitions coincide.

We recall that by [9, Theorem 4] a contraction T is weakly similar to unitary if and only if T is of class C_{11} and its characteristic function Θ_T is (boundedly) invertible a.e. on the unit circle C.

We finish the discussion of weak similarity by the following

Proposition 4. *Weak similarity is an equivalence relation in the class of C_{11}-contractions.*

Proof. We have to verify only transitivity. So let us assume that $T\in\mathcal{L}(\mathfrak{H})$, $S\in\mathcal{L}(\mathfrak{K})$ and $R\in\mathcal{L}(\mathfrak{L})$ are C_{11}-contractions such that T is weakly similar to S and S is weakly similar to R. Then, there exist basic systems $\{\mathfrak{H}_n\}_n$ and $\{\mathfrak{K}_n\}_n$ consisting of hyperinvariant subspaces of T and S, respectively, such that $\mathcal{I}(T|\mathfrak{H}_n, S|\mathfrak{K}_n)$ contains an affinity A_n, for every n. Similarly, we can find basic systems $\{\mathfrak{K}'_n\}_n$ and $\{\mathfrak{L}_n\}_n$ formed by hyperinvariant subspaces of S and R, respectively, such that $\mathcal{I}(S|\mathfrak{K}'_n, R|\mathfrak{L}_n)$ contains an affinity B_n, for every n. Since each of the above subspaces is C_{11}-invariant, and the C_{11}-hyperinvariant subspace lattice of any C_{11}-contraction is countably distributive (cf. [8, Proposition 2]), we can easily verify that the subspaces

$$\mathfrak{K}''_n = \bigvee_{i=1}^{n} (\mathfrak{K}_i \overset{(1)}{\bigcap} \mathfrak{K}'_{n+1-i})\in \operatorname{Hyplat}_1 S, \quad n = 1, 2, \ldots,$$

form a basic system in $\mathfrak{K}$. (Cf. also [9, Lemma 7].)

It follows immediately that the system

$$\{\mathfrak{H}'_n = \bigvee_{i=1}^{n} A_i^{-1}(\mathfrak{K}_i \overset{(1)}{\bigcap} \mathfrak{K}'_{n+1-i})\}_n$$

will be basic in $\mathfrak{H}$. Taking into account that the commutant $\{S\}'$ splits into the direct sum $\{S\}'=\{S|\mathfrak{K}_i\}'+\{S|\bigvee_{j\neq i}\mathfrak{K}_j\}'$ we infer that $\mathfrak{K}_i\overset{(1)}{\bigcap}\mathfrak{K}'_{n+1-i}\in\operatorname{Hyplat}_1(S|\mathfrak{K}_i)$. This implies that $A_i^{-1}(\mathfrak{K}_i\overset{(1)}{\bigcap}\mathfrak{K}'_{n+1-i})\in\operatorname{Hyplat}_1(T|\mathfrak{H}_i)$, and in virtue of the splitting $\{T\}'=$ $=\{T|\mathfrak{H}_i\}'+\{T|\bigvee_{j\neq i}\mathfrak{H}_j\}'$ we conclude $A_i^{-1}(\mathfrak{K}_i\overset{(1)}{\bigcap}\mathfrak{K}'_{n+1-i})\in\operatorname{Hyplat}_1 T$. Since this holds for every $1\leq i\leq n$, we obtain that $\mathfrak{H}'_n\in\operatorname{Hyplat}_1 T$, for every n.

Similarly, we can prove that the subspaces $\mathfrak{L}'_n=\bigvee_{i=1}^{n} B_i(\mathfrak{K}_i\overset{(1)}{\bigcap}\mathfrak{K}'_{n+1-i})\in\operatorname{Hyplat}_1 R$, $n=1, 2, \ldots$, form a basic system in $\mathfrak{L}$. Since $T|\mathfrak{H}'_n$ is obviously similar to $R|\mathfrak{L}'_n$, for every n, we get that T and R are weakly similar.

2. Reflexivity of contractions, weakly similar to unitaries

Our main result is the following

Theorem 5. *Let T be a c.n.u. contraction which is weakly similar to unitary. If there exists a function $f\in(\Delta_T L^2(\mathfrak{D}_T))^-$ such that*

$$\int_C \log\|\Theta_T(e^{it})f(e^{it})\|\,dm(t) > -\infty, \tag{1}$$

then

(i) $H^\infty(T)=\operatorname{Alg} T\neq\{T\}''$,

(ii) $\operatorname{Lat} T\neq\operatorname{Lat}_1 T$, *and*

(iii) *T is reflexive.*

This theorem is a generalization of WU's results (cf. [15, Theorem 3] and [16, Theorem 3.8]), who considered c.n.u. C_{11}-contractions with finite defect indeces, and is a counterpart of [9, Theorem 9], which is connected with contractions whose characteristic function is isometric on a subset of positive measure of the unit circle.

The assumption $\int\limits_C \log\|\Theta_T(e^{it})f(e^{it})\|\,dm(t)>-\infty$ $(f\in(\Delta_T L^2(\mathfrak{D}_T))^-)$, occurring in our theorem, implies that $f(e^{it})\neq 0$ a.e. on C. Hence $\operatorname{rank}\Delta_T(e^{it})\geqq 1$, i.e. $\Theta_T(e^{it})$ is not isometric a.e. on C.

Conversely, let us assume that, for the c.n.u. C_{11}-contraction T, $\Theta_T(e^{it})$ is not isometric a.e. on C. It follows that $\operatorname{rank}\Delta_T(e^{it})\geqq 1$ a.e., and so the operator R of multiplication by e^{it} on the space $(\Delta_T L^2(\mathfrak{D}_T))^-$ is unitarily equivalent to an operator of the form $\bigoplus\limits_n M_{\alpha_n}$, where $C=\alpha_1\supset\alpha_2\supset\dots$ are Borel subsets of C and M_{α_n} denotes the multiplication operator by e^{it} on the space $L^2(\alpha_n, m)$. (Cf. [7, Lemma 1].) This implies that we can find a vector $f\in(\Delta_T L^2(\mathfrak{D}_T))^-$ such that $R|\bigvee\limits_{n\geqq 0} R^n f\in C_{10}$. Then we infer by Lemma 9 to be proved later that

$$\int\limits_C \log\|f(e^{it})\|\,dm(t)>-\infty.$$

Let us assume in addition that Θ_T has a scalar multiple. On account of [12, Proposition V.7.1] this happens exactly when

$$\int\limits_C \log\|\Theta_T(e^{it})^{-1}\|\,dm(t)<\infty.$$

Hence, taking into account that

$$\|\Theta_T(e^{it})f(e^{it})\|\geqq\|\Theta_T(e^{it})^{-1}\|^{-1}\|f(e^{it})\|,$$

we obtain

$$\int\limits_C \log\|\Theta_T(e^{it})f(e^{it})\|\,dm(t)>-\infty.$$

Since in virtue of [9, Remark 5] T is in particular weakly similar to unitary, the assumptions of our theorem are fulfilled.

Therefore, taking also into consideration [9, Theorem 9 and Corollary 12] and that the question of reflexivity can be reduced to the case of c.n.u. contractions (cf. the proof of [2, Theorem 5]), we obtain the following

Corollary 6. *If T is a C_{11}-contraction whose characteristic function Θ_T has a scalar multiple, then T is reflexive. If we assume in addition that T is c.n.u. and*

$\Theta_T(e^{it})$ is not isometric a.e. on C, then

$$H^\infty(T) = \operatorname{Alg} T \neq \{T\}'' \quad \textit{and} \quad \operatorname{Lat} T \neq \operatorname{Lat}_1 T,$$

while if $\Theta_T(e^{it})$ is isometric on a set of positive measure, then

$$H^\infty(T) \neq \operatorname{Alg} T = \{T\}'' \quad \textit{and} \quad \operatorname{Lat} T = \operatorname{Lat}_1 T.$$

The proof of Theorem 5 follows the general outline of Wu's proof in [15]. The framework is the functional model of c.n.u. contractions. So we are starting by recalling some basic facts on model-operators. Since we are interested only in C_{11}-contractions we may restrict attention to contractive analytic functions whose values are operators acting in one Hilbert space.

So let us given a purely contractive analytic function $\{\Theta(\lambda), \mathfrak{E}, \mathfrak{E}\}$, where $\mathfrak{E}$ is a separable Hilbert space and $\Theta(\lambda)\in\mathcal{L}(\mathfrak{E})$ for every $\lambda\in\mathbf{D}$. The model-operator associated with Θ is defined in the following way. Let Δ denote the measurable operator-valued function defined by $\Delta(e^{it})=[I-\Theta(e^{it})^*\Theta(e^{it})]^{1/2}$, and let us consider the Hilbert space

$$\mathfrak{K}_+ = H^2(\mathfrak{E})\oplus(\Delta L^2(\mathfrak{E}))^-$$

of vector-valued functions. The operator $V\in\mathcal{L}(H^2(\mathfrak{E}), \mathfrak{K}_+)$, $Vw=\Theta w\oplus\Delta w$ ($w\in H^2(\mathfrak{E})$) will be an isometry, and the subspace $VH^2(\mathfrak{E})$ of $\mathfrak{K}_+$ will be invariant under the operator U_+ of multiplication by e^{it} in $\mathfrak{K}_+$. Then the model-space is by definition

$$\mathfrak{H} = \mathfrak{K}_+ \ominus VH^2(\mathfrak{E}),$$

and the model-operator $T=S(\Theta)$ is the compression of U_+ onto $\mathfrak{H}$:

$$T = P_{\mathfrak{H}} U_+|\mathfrak{H},$$

where $P_{\mathfrak{H}}$ denotes the orthogonal projection of $\mathfrak{K}_+$ onto $\mathfrak{H}$.

U_+ will be the minimal isometric dilation of T. The subspace $\mathfrak{R}=(\Delta L^2(\mathfrak{E}))^-$ reduces U_+ to a unitary operator $R=U_+|\mathfrak{R}$, called the residual part of T. Since $VH^2(\mathfrak{E})\in\operatorname{Lat} U_+$, it follows that $P_{\mathfrak{H}}U_+=TP_{\mathfrak{H}}$, and so the operator

(2) $$Y = P_{\mathfrak{H}}|\mathfrak{R}(= (P_{\mathfrak{R}}|\mathfrak{H})^*)$$

intertwines R and T: $Y\in\mathcal{I}(R, T)$. Moreover, on account of [12, Proposition II.3.5] Y is a quasi-affinity if T is a C_{11}-contraction.

By the Lifting Theorem there is a close connection between the commutants of U_+ and T (cf. [12, Theorem II.2.3] and [13]). Namely, let us denote by $\{U_+\}'_0$ the set of those operators in the commutant of U_+ which leave invariant the subspace $VH^2(\mathfrak{E})$: $\{U_+\}'_0=\{\hat{Q}\in\{U_+\}': \hat{Q}VH^2(\mathfrak{E})\subset VH^2(\mathfrak{E})\}$. Then the Lifting Theorem says that the mapping

$$\pi\colon \{U_+\}'_0 \to \{T\}', \quad \pi\hat{Q} = P_{\mathfrak{H}}\hat{Q}|\mathfrak{H} \quad (\hat{Q}\in\{U_+\}'_0)$$

will be a well-defined, contractive, surjective, algebra-homomorphism.

Let us consider the matrix of an arbitrary operator $\hat{Q}\in\{U_+\}_0'$ with respect to the decomposition $\mathfrak{K}_+=H^2(\mathfrak{E})\oplus\mathfrak{R}$:

$$\hat{Q}=\begin{bmatrix} A & D \\ B & C \end{bmatrix}.$$

Since $\hat{Q}$ commutes with U_+, it follows that $D\in\mathscr{I}(R, U_+|H^2(\mathfrak{E}))$. Taking into account that R is unitary and $\bigcap_{n\geq 0} U_+^n H^2(\mathfrak{E})=\{0\}$, we deduce that $D=0$, i.e. $\mathfrak{R}\in\operatorname{Lat}\hat{Q}$. The relation $\hat{Q}VH^2(\mathfrak{E})\subset VH^2(\mathfrak{E})$ implies that $P_{\mathfrak{H}}\hat{Q}=QP_{\mathfrak{H}}$, where $Q=\pi\hat{Q}$. Hence, considering restrictions onto the subspace $\mathfrak{R}$, we obtain

$$YC=QY, \tag{3}$$

where C commutes with R.

Now we prove a lemma on the model-operator T introduced above.

Lemma 7. *Let us given* $\hat{Q}=\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}\in\{U_+\}_0'$ *and* $Q=\pi\hat{Q}\in\{T\}'$. *If* T *is a* C_{11}-*contraction, then* $Q=0$ *is equivalent to* $C=0$, *and* $C\in\{R\}''$ *implies* $Q\in\{T\}''$. *Moreover, if* T *is weakly similar to unitary, then* $C\in\{R\}''$ *and* $Q\in\{T\}''$ *are equivalent.*

Proof. If T is a C_{11}-contraction, then the operator Y defined in (2) is a quasi-affinity. Hence the intertwining relation (3) yields that Q and C are equal to zero simultaneously.

Let us assume that $C\in\{R\}''$, and let us consider an arbitrary operator $Q'\in\{T\}'$. Since the mapping π is surjective, we can find an operator $\hat{Q}'=\begin{bmatrix} A' & 0 \\ B' & C' \end{bmatrix}\in\{U_+\}_0'$ such that $\pi\hat{Q}'=Q'$. In virtue of our assumption $C\in\{R\}''$ it follows that the operator $\hat{Q}'':=\hat{Q}\hat{Q}'-\hat{Q}'\hat{Q}\in\{U_+\}_0'$ has a matrix of the form

$$\begin{bmatrix} A'' & 0 \\ B'' & CC'-C'C \end{bmatrix}=\begin{bmatrix} A'' & 0 \\ B'' & 0 \end{bmatrix}.$$

Therefore, on account of the first part of our lemma, proved before, we conclude that $\pi\hat{Q}''=0$. However, π being an algebra-homomorphism this yields that $0=\pi\hat{Q}''=$ $=QQ'-Q'Q$, i.e. Q commutes with Q'.

Let us assume now that T is weakly similar to unitary, $Q\in\{T\}''$, and let us consider an arbitrary operator $C'\in\{R\}'$. On account of [9, Theorem 4] T belongs to C_{11} and $\Theta_T(e^{it})$ is boundedly invertible a.e. on C. Let $\alpha_n\subset C$ be the measurable set $\alpha_n=\{e^{it}\colon \|\Theta(e^{it})^{-1}\|\leq n\}$, for every n. Then $\{\alpha_n\}_n$ forms an increasing sequence such that $m(C\setminus(\bigcup_n \alpha_n))=0$. Consequently, if χ_{α_n} denotes also the operator of multiplication by the characteristic function χ_{α_n} of α_n, then the sequence $\{\chi_{\alpha_n}\}_n\subset\{R\}''$ tends to the identity operator $I_{\mathfrak{R}}$ in the strong operator topology.

For every n, let $\hat{Q}_n \in \{U_+\}'$ denote the operator whose matrix in the decomposition $\mathfrak{K}_+ = H^2(\mathfrak{E}) \oplus \mathfrak{R}$ is the following

$$\hat{Q}_n = \begin{bmatrix} 0 & 0 \\ -C'(\chi_{\alpha_n} \Delta \Theta^{-1}) & \chi_{\alpha_n} C' \end{bmatrix}.$$

Here $\chi_{\alpha_n} \Delta \Theta^{-1}$ stands for the operator of multiplication by the bounded, measurable, operator-valued function $\chi_{\alpha_n} \Delta \Theta^{-1}$. Since for every $w \in H^2(\mathfrak{E})$ we have

$$\hat{Q}_n V w = \hat{Q}_n (\Theta w \oplus \Delta w) = 0 \oplus (-C' \chi_{\alpha_n} \Delta \Theta^{-1} \Theta w + \chi_{\alpha_n} C' \Delta w) = 0 \oplus 0,$$

it follows that $\hat{Q}_n \in \{U_+\}'_0$. Hence $Q_n = \pi \hat{Q}_n \in \{T\}'$, and so $Q_n Q = Q Q_n$, for every n. In virtue of the first part of our lemma we conclude that

$$\chi_{\alpha_n}(C'C) = (\chi_{\alpha_n} C')C = C(\chi_{\alpha_n} C') = \chi_{\alpha_n}(CC')$$

holds, for every n. Taking into account that $\{\chi_{\alpha_n}\}_n$ converges to the identity, we obtain that

$$C'C = CC'.$$

Therefore, C belongs to $\{R\}''$, and so the proof is completed.

In order to formulate our second lemma on the model-operator T we introduce the operator-valued function $\Delta_*(e^{it}) = [I - \Theta(e^{it})\Theta(e^{it})^*]^{1/2}$. Then the operator R_*, called the $*$-residual part of T, is defined as the multiplication by e^{it} on the Hilbert space $\mathfrak{R}_* = (\Delta_* L^2(\mathfrak{E}))^-$. The following lemma, which is a generalization of [16, Lemma 3.4] (cf. also [11]), is proved in [10].

Lemma 8. *If T is a C_{11}-contraction, then the mapping*

$$X\colon \mathfrak{H} \to \mathfrak{R}_*, \quad X(u \oplus v) = -\Delta_* u + \Theta v \quad (u \oplus v \in \mathfrak{H}), \tag{4}$$

is a (well-defined) quasi-affinity, belonging to $\mathscr{I}(T, R_*)$. Moreover, its product $Z = XY \in \mathscr{I}(R, R_*)$ with the operator Y, defined in (2), *acts as a multiplication by Θ, i.e.*

$$(Zv)(e^{it}) = \Theta(e^{it}) v(e^{it})$$

holds a.e. on C, for every $v \in \mathfrak{R}$.

Finally, we need two lemmas concerning absolutely continuous unitary operators.

Lemma 9. *Let U be the operator of multiplication by e^{it} on the space $\mathfrak{K} = L^2(\mathfrak{F})$, where $\mathfrak{F}$ is a Hilbert space, and for any non-zero vector $h \in \mathfrak{K}$ let $\mathfrak{K}_h$ denote the invariant subspace $\mathfrak{K}_h = \bigvee_{n \geq 0} U^n h$. Then the restriction $U|\mathfrak{K}_h$ belongs to the class C_{10} if and only if*

$$\int_C \log \|h(e^{it})\| \, dm(t) > -\infty.$$

Proof. Let $\mathfrak{K}_{h,0}$ denote the linear manifold $\mathfrak{K}_{h,0}=\{p(U)h: p(\lambda)$ is a complex polynomial$\}$, and let us define the mapping $V_0: \mathfrak{K}_{h,0}\to L^2(C, m)$ by $(V_0(p(U)h))(e^{it})= =p(e^{it})\|h(e^{it})\|_{\mathfrak{F}}$. It is immediate that V_0 is a (linear) isometry (hence well-defined), and so it can be extended to an isometry $V\in\mathcal{L}(\mathfrak{K}_h, L^2(C, m))$. Since we evidently have $V_0(U|\mathfrak{K}_{h,0})=MV_0$, where M denotes the operator of multiplication by e^{it} in $L^2(C, m)$, it follows that

$$V(U|\mathfrak{K}_h) = MV.$$

This yields that $\operatorname{ran} V\in\operatorname{Lat} M$ and $U|\mathfrak{K}_h$ is unitarily equivalent to $M|\operatorname{ran} V$. Therefore $U|\mathfrak{K}_h$ belongs to the class C_{10} if and only if so does the operator $M|\operatorname{ran} V$. However, taking into account that

$$\operatorname{ran} V = \bigvee_{n\geqq 0} M^n\|h\|,$$

we conclude that $M|\operatorname{ran} V\in C_{10}$ holds exactly when

$$\int_C \log\|h(e^{it})\|\,dm(t) > -\infty.$$

(Cf. the Szegő—Kolmogoroff—Krein theorem in [6].)

Lemma 10. *Let $U\in\mathcal{L}(\mathfrak{K})$ be an absolutely continuous unitary operator, and let us consider an operator $C\in\{U\}''$. If C leaves invariant a non-zero subspace $\mathfrak{M}\in\operatorname{Lat} U$ such that $U|\mathfrak{M}\in C_{10}$, then C is of the form $C=\delta(U)$, where δ is a function from H^∞.*

Proof. Since $C\in\{U\}''$, we infer by the spectral theorem (cf. [4]) that C has the form $C=\delta(U)$ with an appropriate function $\delta\in L^\infty(m)$.

The assumption $U|\mathfrak{M}\in C_{10}$ implies that $U|\mathfrak{M}$ is a unilateral shift. Consequently, the subspace $\mathfrak{L}=\mathfrak{M}\ominus U\mathfrak{M}$ is wandering for U, i.e. the sequence $\{U^n\mathfrak{L}\}_{n=-\infty}^{\infty}$ consists of pairwise orthogonal subspaces. Let us consider the subspace

$$\hat{\mathfrak{M}} = \bigoplus_{n=-\infty}^{\infty} U^n\mathfrak{L},$$

which clearly reduces U. Taking into account that $C\in\{U\}''$ we conclude that $\hat{\mathfrak{M}}\in\operatorname{Lat} C$ and

$$C|\hat{\mathfrak{M}} = \delta(U)|\hat{\mathfrak{M}} = \delta(U|\hat{\mathfrak{M}}).$$

Hence, we obtain that

$$\delta(U|\hat{\mathfrak{M}})\mathfrak{M}\subset\mathfrak{M}. \tag{5}$$

Let us consider now the Fourier-representation of $\hat{\mathfrak{M}}$, i.e. the unitary map

$$\Phi\colon \hat{\mathfrak{M}}\to L^2(\mathfrak{L}),\quad \Big(\Phi\Big(\bigoplus_{n=-\infty}^{\infty} U^n h_n\Big)\Big)(e^{it}) = \sum_{n=-\infty}^{\infty} e^{itn}h_n$$

($h_n \in \mathfrak{L}$, for every n). Φ intertwines $U|\hat{\mathfrak{M}}$ with the operator $M\in\mathscr{L}(L^2(\mathfrak{L}))$ of multiplication by e^{it}: $\Phi(U|\hat{\mathfrak{M}})=M\Phi$. This yields the relation

$$\Phi\delta(U|\hat{\mathfrak{M}}) = \delta(M)\Phi. \tag{6}$$

Consequently, on account of (5) and (6) we infer

$$\delta(M)H^2(\mathfrak{L}) \subset H^2(\mathfrak{L}),$$

which implies that $\delta\in H^\infty$, and the proof is finished.

Now we are ready to prove our main theorem.

Proof of Theorem 5. It is enough to show that $\mathrm{Alg\,Lat}\, T\subset H^\infty(T)$. Indeed, then on account of the relations $H^\infty(T)\subset \mathrm{Alg}\, T\subset \mathrm{Alg\,Lat}\, T$ it follows that

$$H^\infty(T) = \mathrm{Alg}\, T = \mathrm{Alg\,Lat}\, T,$$

hence T is reflexive. Moreover, in virtue of [9, Corollary 12] we obtain $\mathrm{Alg}\, T\neq\{T\}''$, and taking into consideration that $\mathrm{Alg\,Lat}_1 T=\{T\}''$ (cf. the proof of [9, Proposition 13]) we conclude $\mathrm{Lat}\, T\neq \mathrm{Lat}_1 T$.

So let $Q\in \mathrm{Alg\,Lat}\, T$ be an arbitrary operator. We shall show that $Q\in H^\infty(T)$. On account of [12, Theorem VI.2.3] we may assume that T is a model-operator $T=S(\Theta)$, where $\{\Theta(\lambda), \mathfrak{E}, \mathfrak{E}\}$ is a purely contractive, analytic function, outer from both sides.

Since Q clearly belongs to $\mathrm{Alg\,Lat}''\, T$, we infer by the reflexivity of $\{T\}''$ (cf. [14]) that

$$Q\in\{T\}''.$$

On account of the Lifting Theorem there is an operator $\hat{Q}=\begin{bmatrix} A & 0\\ B & C\end{bmatrix}\in\{U_+\}_0'$ such that $Q=\pi\hat{Q}$. An application of Lemma 7 gives that

$$C\in\{R\}''.$$

In order to be able to apply Lemma 10 we have to show that $C\mathfrak{M}\subset\mathfrak{M}$ for a non-zero subspace $\mathfrak{M}\in \mathrm{Lat}\, R$ such that $R|\mathfrak{M}\in C_{10}$.

By the assumption there exists a vector $f\in\mathfrak{R}$ such that

$$\int_C \log\|g(e^{it})\|\, dm(t) > -\infty$$

for the function $g=\Theta f$. Now, on account of Lemma 8 we know that g is contained in $\mathfrak{R}_*$. Moreover, applying Lemma 9 we obtain that

$$R_*|\mathfrak{R}_{*,g}\in C_{10}, \tag{7}$$

for the subspace $\mathfrak{R}_{*,g}=\bigvee_{n\geq 0} R_*^n g\in \mathrm{Lat}\, R_*$. Then the intertwining relation $R_*X=XT$, where X is the operator defined in (4), implies that the subspace $\mathfrak{L}=X^{-1}\mathfrak{R}_{*,g}$ is invariant for T. Since, by Lemma 8, X is injective, we see that $T|\mathfrak{L}$ can be injected into $R_*|\mathfrak{R}_{*,g}$:

$$T|\mathfrak{L} \overset{i}{\prec} R_*|\mathfrak{R}_{*,g}. \tag{8}$$

We conclude by (7) and (8) that the operator $T|\mathfrak{L}$ is also of class C_{10}.

An analogous argumentation yields that the subspace

$$\mathfrak{M} = Y^{-1}\mathfrak{L},$$

where Y is defined by (2), is invariant for R and

$$R|\mathfrak{M} \in C_{10}.$$

Since the non-zero vector f clearly belongs to $\mathfrak{M}$, it follows that $\mathfrak{M}$ is non-zero. On the other hand, $\mathfrak{L} \in \operatorname{Lat} T$ and $Q \in \operatorname{Alg\,Lat} T$ imply

$$\mathfrak{L} \in \operatorname{Lat} Q.$$

Hence, the intertwining relation (3) yields that

$$\mathfrak{M} \in \operatorname{Lat} C.$$

Now, we can apply Lemma 10 to obtain that C has the form $C = \delta(R)$, with a suitable function $\delta \in H^\infty$.

Since the operator Y intertwines R and T too, we infer

$$\delta(T)Y = Y\delta(R) = YC.$$

Comparing this equality with (3) we conclude that

$$\delta(T)Y = QY.$$

Consequently, taking into account that Y is a quasi-affinity we obtain

$$Q = \delta(T).$$

The theorem is proved.

3. Concluding remarks

Under more general assumptions we are able to prove the following weaker version of part (i) of Theorem 5.

Proposition 11. *If T is a c.n.u. C_{11}-contraction such that $\Theta_T(e^{it})$ is not isometric a.e. on C, then*

$$H^\infty(T) = \operatorname{Alg}_* T,$$

where $\operatorname{Alg}_* T$ denotes the algebra generated by T and the identity, and closed in the ultraweak operator topology.

Proof. First of all we note the elementary fact that if an operator S is similar to a normal operator N, then $\|S\| \geqq \|N\|$. Indeed, similarity preserves the spectrum, so if r_S, r_N denote the spectral radii of S and N, respectively, then we can write $\|N\| = r_N = r_S \leqq \|S\|$.

Let us assume now that $T \in \mathscr{L}(\mathfrak{H})$ is a c.n.u. C_{11}-contraction such that $\Theta_T(e^{it})$ is not isometric a.e. on C. We can find an absolutely continuous unitary operator $U \in \mathscr{L}(\mathfrak{K})$, which is quasi-similar to T (cf. [12, Proposition II.3.5 and Theorem II.6.4]). By a result of Apostol (cf. [1]) there are basic systems $\{\mathfrak{H}_n\}_n$ and $\{\mathfrak{K}_n\}_n$ in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, such that $\mathfrak{H}_n \in \operatorname{Lat} T$, $\mathfrak{K}_n \in \operatorname{Lat} U$ and $T|\mathfrak{H}_n$ is similar to $U|\mathfrak{K}_n$, for every n. Moreover, it can be achieved that the subspaces $\{\mathfrak{K}_n\}_n$ are pairwise orthogonal, i.e. the decomposition $\mathfrak{K} = \bigoplus_n \mathfrak{K}_n$ reduces U.

Let us given an arbitrary function $w \in H^\infty$. Since $w(T|\mathfrak{H}_n)$ is similar to the normal operator $w(U|\mathfrak{K}_n)$, we infer that

$$\|w(T)\| \geqq \|w(T)|\mathfrak{H}_n\| = \|w(T|\mathfrak{H}_n)\| \geqq \|w(U|\mathfrak{K}_n)\| = \|w(U)|\mathfrak{K}_n\|,$$

for every n, hence

$$\|w(T)\| \geqq \sup_n \|w(U)|\mathfrak{K}_n\| = \|w(U)\|.$$

However, $\Theta_T(e^{it})$ being not isometric a.e. on C, it follows by [7, Corollary 1] and [12, Proposition II.3.4] that $\sigma(U) = C$, and so $\|w(U)\| = \|w\|_\infty$. Therefore, we conclude that $\|w(T)\| \geqq \|w\|_\infty$. Since the opposite direction always holds (cf. [12, Theorem III.2.1]), we obtain that

$$\|w(T)\| = \|w\|_\infty,$$

for every $w \in H^\infty$, i.e. the Sz.-Nagy, Foiaş functional calculus is an isometry. But then on account of [3, Theorem 3.2] we get that

$$H^\infty(T) = \operatorname{Alg}_* T,$$

and the proof is finished.

It is left open whether the statements of Theorem 5 remain true under the assumption of Proposition 11, even in the case when T is weakly similar to unitary. The following example illuminates where difficulties arise.

Example 12. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint Borel subsets of the unit circle C such that $m(\alpha_n) > 0$, for every n, and $\sum_n m(\alpha_n) = 1$. Let us choose an arbitrary sequence $\{c_n\}_{n=1}^\infty$ of positive numbers, where $c_n < 1$ for every n, and for each n let us define a (scalar-valued) outer function ϑ_n by the boundary condition

$$|\vartheta_n(e^{it})| = c_n \chi_{\alpha_n}(e^{it}) + \chi_{C \setminus \alpha_n}(e^{it}) \quad \text{a.e.}$$

Let us consider a separable, infinite dimensional Hilbert space $\mathfrak{E}$, and an orthonormal basis $\{e_n\}_n$ in $\mathfrak{E}$. Then $\{\Theta(\lambda), \mathfrak{E}, \mathfrak{E}\}$ will stand for the contractive, operator-valued, analytic function, whose matrix is

$$[\Theta(\lambda)] = \operatorname{diag}(\vartheta_n(\lambda)), \quad \lambda \in \mathbf{D},$$

in this basis. We shall examine the model-operator

$$T = S(\Theta),$$

which, of course, depends on the choice of sequences $\{\alpha_n\}_n$ and $\{c_n\}_n$.

Since Θ is outer from both sides, it follows that T is of class C_{11}. Moreover, the identity

$$\|\Theta(e^{it})^{-1}\| = \sum_n \chi_{\alpha_n}(e^{it})c_n^{-1},$$

being valid a.e. on C, implies that T is weakly similar to unitary (cf. [9, Theorem 4]).

Since $(\Delta L^2(\mathfrak{E}))^-$ splits into the orthogonal sum $(\Delta L^2(\mathfrak{E}))^- = \bigoplus_n (\Delta_n L^2(\mathfrak{E}_n))^-$, where $\mathfrak{E}_n$ is the one-dimensional subspace of $\mathfrak{E}$ spanned by e_n, $\Delta_n(e^{it})$ acts on $\mathfrak{E}_n$, and $\Delta_n(e^{it})e_n = (1-|\vartheta_n(e^{it})|^2)^{1/2}e_n$, it follows easily that relation (1) in Theorem 5 is satisfied with a vector $f \in (\Delta L^2(\mathfrak{E}))^-$ if and only if Θ has a scalar multiple, i.e. if

$$\infty > \int_C \log \|\Theta(e^{it})^{-1}\|\, dm(t) = \sum_n (\log c_n^{-1})\, m(\alpha_n)$$

holds. Hence, Theorem 5 can be applied exactly when

$$\sum_n m(\alpha_n) \log c_n^{-1} < \infty. \tag{9}$$

Let us examine now what the spectrum $\sigma(T)$ of T is like. We know by [12, Theorem VI.4.1] that a point μ of the unit disc $\mathbf{D}$ belongs to the spectrum if and only if $\Theta(\mu)$ is not invertible, which is equivalent to the condition

$$\sup_n |\vartheta_n(\mu)|^{-1} = \infty.$$

Taking into account that

$$\exp\left[\frac{1-|\mu|}{1+|\mu|} \log c_n^{-1} \cdot m(\alpha_n)\right] \leqq |\vartheta_n(\mu)|^{-1} = \exp\Big[\int_{\alpha_n} P_r(\varphi - t) \log c_n^{-1}\, dm(t)\Big] \leqq$$

$$\leqq \exp\left[\frac{1+|\mu|}{1-|\mu|} \log c_n^{-1} \cdot m(\alpha_n)\right],$$

where $\mu = re^{i\varphi}$, and P_r denotes the Poisson-kernel, we infer that $\mu \in \sigma(T)$ exactly when the equality

$$\sup_n [m(\alpha_n) \log c_n^{-1}] = \infty, \tag{10}$$

independent of μ, holds. Since $\sigma(T)$ always includes the whole unit circle C (cf. [12, Theorem VI.4.1]), we obtain that $\sigma(T)=\mathbf{D}^-$ or $\sigma(T)=C$ according to the case when (10) is fulfilled or not.

Taking into consideration that the essential spectrum $\sigma_e(T)$ of the C_{11}-contraction T coincides with its spectrum, we conclude that $\sigma_e(T)$ is dominating in $\mathbf{D}$, i.e. T is a (BCP)-operator (cf. [2]) if and only if (10) holds. But then [2, Theorem 1] also yields that the statements of Theorem 5 are true. (Cf. also the beginning of the proof of Theorem 5.)

Summerizing, we have obtained that the statements (i)—(iii) of Theorem 5 are valid if (9) or (10) are fulfilled, i.e. either if the sequence $\{m(\alpha_n) \log c_n^{-1}\}_n$ tends to zero fast enough or if it is unbounded. The intermediate case remains open.

Added in proof (December 10, 1987). In a subsequent paper, appearing in *Acta Math. Hung.* **50** (1987), further developing the methods of this work we succeeded in answering the question raised above.

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